Advanced Math. Models & Applications, V.1, N.1, 2016, pp.50-55

ON AN OPTIMIZATION PROBLEM FOR THE EIGENVALUES

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Abstract. We investigate one shape optimization problem involving the first eigenvalue of the biharmonic operator, describing the transverse vibrations of the clamped plate. Taking domain of the plate variable we consider its eigenfrequency as a domain functional and investigate it.

Keywords: Clamped plate, transverse vibrations, eigenfrequency, domain functional, domain vibration, support function.

AMS Subject Classification: 49J45, 49Q10, 49R50.

1. Introduction

In continuum mechanics, plate theories are mathematical descriptions of the mechanics of flat plates that draws on the theory of beams. Plates are the plane structural elements with a small thickness compared to the planar dimensions. A plate theory uses this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem. From the existing plate theories two are widely accepted and used in engineering. These are the Kirchhoff-Love theory of plates (classical plate theory) [9] and the Mindlin-Reissner theory of plates (first order shear plate theory) [8]. In both these theories the area (domain) of the plate is fixed and the aim is to calculate the deformation and stresses in a plate subjected to loads [6]. But some practical situations put a problem to find and then optimize (minimize or maximize) the eigenfrequency of the plate under across vibrations [7]. The optimization may be done by choosing the physical characteristics of the plate. Some engineering solutions require the optimization of the eigenfrequency by varying the form (area, domain) of the plate. For these problems it is expedient to consider a plate with non-fixed, variable area. Then the eigenfrequency of such plate may be considered as functional depending on the plate area (domain). By this way we arrive to the shape optimization problems [3]. Note that the existence in such problems is investigated by various authors [4, 5]. As is shown these problem are well-posed if the set of admissible domains satisfy some geometrical restrictions, for example, are open sets, or the functional under minimization depends on lower number of eigenvalues (in the case of plates-eigenfrequences).

In this work we consider a plate with variable domain and study its eigenfrequency as a domain functional. Note that we use the approach proposed in [10] that gives new definition of the variation of the domain and then offers a scheme for calculation of the first variation of the functional with respect to domain. This approach allows one to study a wide class of similar problems from theoretical and practical points of view [10].

Note that, given in [10] approach then was extended for lager classes in functions by some authors. For instance in [1] the shape derivative formula for the integral cost functional with respect to a class of admissible convex domains given in [10] is extended to the case of $W^{1,1}_{loc}$ functions. In [2] the obtained results are implemented in Brunn-Minkowski theory.

2. Formulation of the problem and main results

Consider the following problem

$$\Delta^2 u = \lambda u, \quad x \in D, \tag{1}$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad x \in S_D, \tag{2}$$

where Δ is Laplace operator, $D \subset R^n$ is a bounded convex domain with smooth boundary $S_D \in C^2$. The set of such domains denote by K.

It is known that the equation (1) with boundary condition (2) describes the transverse vibrations of the clamped plate. λ in (1) is an eigenvalue of the operator Δ^2 and indeed describes the eigenfrequency of this plate [11].

The object under investigation is the following minimization problem subject to (1), (2)

$$J(D) = \lambda_1(D) + \int_D f(x)dx \to \min, \tag{3}$$

where $\lambda_1(D)$ is the first eigenvalue (i.e. the first eigenfrequency of the clamped plate) of the problem (1), (2) corresponding to $D \subset R^n$, $f(x)$ is given continuously differentiable in R^n function.

Note that investigation of some applied problems lead to the studying the functionals type (3) that put relation between eigenfrequency of the plate and some mechanical parameters, as well as external influence or physical characteristics of the plate.

The main result of the work is the following

Theorem. If the domain $D \subset K$ is a solution of the problem (1)-(3), then

$$\lambda_1(D) = \frac{1}{4} \int_{S_D} f(x) P_D(n(x)) ds \ . \tag{4}$$

Here

$$P_D(x) = \sup_{l \in D}(l, x), x \in R^n$$

is a support function of the domain $D \subset R^n$, $n(x)$ is outward normal to the boundary S_D in the point x, s is boundary element.

Proof. It is known that for fixed D the first eigenvalue of the problem (1), (2) is calculated by formula [11]

$$\lambda_1(D) = \inf_u I(u, D), \tag{5}$$

where

$$I(u, D) = \frac{\int_D (\Delta u(x))^2 dx}{\int_D u^2(x) dx}, \quad |\nabla u(x)|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2$$

and *inf* is taken over all functions, $u \in C^2(D) \cap C^1(\overline{D})$, being equal to zero at S_D.

As we see, formula (5) defines λ_1 as a functional of D.

In [10] the differentiability of the functional $\lambda_1(D)$ with respect to D on K is proved and the expressions are obtained for its first variation under different boundary conditions. It is known that the first eigenvalue of the problem (1), (2) is simple. In this case for this problem the expression is true

$$\delta \lambda_1(D) = -\int_{S_D} |\Delta u_1(x)|^2 [P_{D'}(n(x)) - P_D(n(x))] ds, \tag{6}$$

where $D, D' \in K$, $n(x)$ is an outward normal to S_D at the point x.

As one may obtain from (6)

$$\lambda_1(t + \Delta t) - \lambda_1(t) = \lambda_1(D(t + \Delta t)) - \lambda_1(D(t)) =$$

$$= \int_{S(t)} |\nabla u_1(x)|^2 [P_{D(t+\Delta t)}(n(x)) - P_{D(t)}(n(x))] ds + o(\Delta t).$$

From this dividing by Δt we get

$$\lambda_1'(t) = -\int_{S_{D(t)}} |u_1(x)|^2 P'_{D(t)}(n(x)) ds, \tag{7}$$

where

$$P'_{D(t)}(x) = \frac{d}{dt} P_{D(t)}(x).$$

Now let's show that for the first eigenvalue of the problem (1), (2) in the domain D is true the formula

$$\lambda_1(D) = \frac{1}{4} \int_{S_D} (u_1(x))^2 P_D(n(x)) ds. \tag{8}$$

Let $D(r) = rD_0, r > 0$. The first eigenfrequency of the clamped plate in the domain D_0 denote by $u_1(x)$. This means that

$$\Delta^2 u_j(x) = \lambda_j(D_0)u_j(x), \quad x \in D_0 . \tag{9}$$

One can write this equation in the following equivalent form

$$\frac{1}{r^4}\Delta_y^2 u_j\left(\frac{x}{r}\right) = \frac{\lambda_j(D_0)}{r^4}u_j\left(\frac{x}{r}\right), \; x \in D(r),$$

where

$$y = \frac{x}{r} .$$

Take

$$\tilde{u}_j(x) = u_j\left(\frac{x}{r}\right), \quad x \in D(r) .$$

Then

$$\Delta^2 \tilde{u}_j(x) = \frac{1}{r^4}\Delta_y^2 u_j\left(\frac{x}{r}\right) .$$

The last shows that $\tilde{u}_j(x)$ is an eigenvibration and $\frac{\lambda_j(D_0)}{r^4}$ - corresponding eigenfrequency of the plate with domain $D(r)$ i.e. $\lambda_j(r) = \frac{\lambda_j(d_0)}{r^4}$. Substituting this into (9), considering $P_{D(r)}(x) = r \cdot P_{D_0}(x)$ and taking $r = 1$we get (8).

In [10] differentiability of the functional

$$F(D) = \int_D f(x)dx \tag{10}$$

is proved under the given conditions and the following formula is obtained for the its first variation

$$\delta F(D) = \int_{S_D} f(x)[P_{D'}(n(x)) - P_D(n(x))]ds . \tag{11}$$

Now, let $D \in K$ be a solution of the problem (1)-(3). Then according to the optimality condition

$$-|u_1(x)|^2 + f(x) = 0 , \quad x \in S_D . \tag{12}$$

Multiplying (12) by $P_D(n(x))$ and integrating over S_D one may get

$$-\frac{1}{4}\int_{S_D}|u_1(x)|^2 P_D(n(x))ds + \frac{1}{4}\int_{S_D} f(x)P_D(n(x))ds = 0 .$$

Considering here (11) we get (4).
Theorem is proved.

Let's consider a particular case. Suppose that $f(x)=1, x\in R^n$. In this case the functional (3) takes a form

$$J(D)=\lambda_1(D)+mesD\to \min . \qquad (10)$$

From (4) we obtain

$$\lambda_1(D)=\frac{1}{4}\int_{S_D} P_D(n(x))ds .$$

In two dimensional case

$$\frac{1}{4}\int_{S_D} P_D(n(x))ds = mesD .$$

Thus

$$\lambda_1(D)=mesD .$$

Note, that when $f(x)=0$ the problem (1)-(3) has no solution. In this case the "increasing" of domain leads to decreasing of the eigenvalue. From (8) also follows the condition $\lambda_1=0$.

Other particular cases also may be considered.

Note that using the obtained formula (6) for the first variation of the eigenvalue of the problem (1), (2) with respect to domain and corresponding optimality condition the numerical algorithm is proposed for the finding of the optimal shape. This algorithm is similar to conditional gradient method. The numerical experiments have been carried out for some functionals of the first eigenvalue of the problem (1), (2), in two dimensional case.

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